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Some Aspects of Chess Game Relating to Knot Theory

S. I. Nada & A. I. Elrokh

Abstract— In this paper we present an analysis for the movement of chess pieces and explain how it relates to the theory of knots. We show that for each piece in the chess except pawns, one can form a sequence of steps to construct a specific knot.

Index Terms— Knots, Cyclic graph, Chess.

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1 Introduction

CHESS is a game with deep historical origins dating back as far as the 6th century AD, and first version of the game emerged in India under the name of Chaturanga. Then the game quickly spread to Persia and subsequently the Arab world. Eventually spread to Europe. Although the game is thought to have differed widely from the game we play today, around the end of the 15th century many of the chessmen developed their modern powers, enhancing the speed and dynamics of the game and creating the game we play now. At the beginning of each game the chessboard is set up in the initial position. Each player begins with 8 pieces (1 King, 1 Queen, 2 Rooks, 2 Bishops and 2 Knights) and 8 pawns which are located in their respective positions.

Knot theory deals with knots, links, braids, and related objects. It is a branch of algebraic topology which started in the late 1880's. The scotch mathematician, Lord William Thomson(1824-1907), proposed that different elements consists of different configurations of knots. He described atoms to be knots in the fabric of this ether. This theory led many Scientists to believe that we could understand the chemical elements by simply studying different types of knots and thus this led to a completely new field of study in Mathematics .One of the most important applications of the knot theory arises in study of DNA[4]. It was discovered in 1953 by James Watson and Francis Crick that the basic structure of DNA molecules consists of two stands which are twisted together in a right-hounded helix. The geometric of DNA can exist in linear form or in closed circular form. It is known that the circular form of DNA can be knotted and two or more circular DNA forms can be linked together. Many other application of Knot theoryin particle physics can be found in [7], [8], statistical mechanics [6], Islamic and Pharaohs Arts[1], and more. [9], [10], [11].The Feynman relativistic chessboard model provides a representation of the solution to Dirc equation in (1+1)-dimension space time of quantum mechanics[ ]. A reformulation of the Feynman chessboard model has been made by Ord [ ]. Also, there is a relation between this model and stochastic model of the Telegraph equation[ ]. It is proved that this mode l was embedded in correlations stochastic model [ ].

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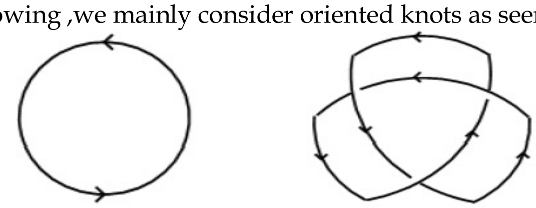
2 Basic Definitions and Terminology

A knot K is a subspace of the three dimensional Euclidean space $\mathbb{R}^3$, which is a topological image of the circle s^1 (as an embedding) $k : s^1 \to \mathbb{R}^3$ of the circle s^1 into three dimensional Euclidean space $\mathbb{R}^3$. Some examples are illustrated down

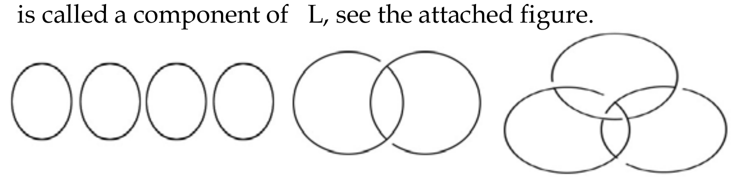
Circle s^1(unknot) trefoil knot some more complicated knot

A knot can be taken with or without orientation. In the following ,we mainly consider oriented knots as seen below



Oriented s̀ oriented trefoil

A link graph, or shortly link, L is an embedding of a topological sum of finitely many copies of a circle s^1 into $\mathbb{R}^3$, $L: s^1 \cup s^1 \ldots \cup s^1 \to \mathbb{R}^3$. The restriction of L to one of the copies of s^1 is called a component of L, see the attached figure.



n-unlink ($s^1 \cup s^1 \cup \ldots \cup s^1$)2-chain Borromean rings

A diagram of a knot or link is a projection of the latter into a plane with marking of each crossing (under-crossing or over crossing) in the image of the projection. That means a knot diagram is picture of a projection of knot onto a plane, a dia-

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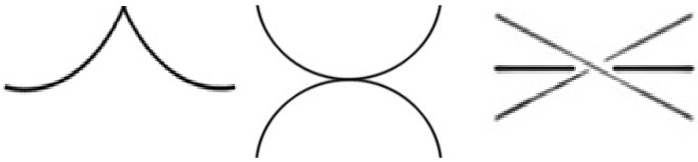
gram in $\mathbb{R}^2$ is made up of a number of arcs and crossing . At a crossing, one arc is over pass and the other is under pass, and it is not allowed the following cases :



Graph of a diagram of a knot or link is the graph consisting of the crossings as the vertices of the graphs and the edges are the arcs between two each crossings. In other words: A graph Γ is a link graph (knot-graph) if there is a link $\mathbf{L}$ (knot $\mathbf{K}$) such that suitable projection of $\mathbf{L}$ ($\mathbf{K}$) which gives Γ. It is denoted by $\Gamma(L)(\Gamma(K))$.

Two graphs $\mathbf{G}$ and H are said to be isomorphic (written G $\sim$H) if there are two bijections $\theta: v(G) \rightarrow v(H)$ and $\varphi: E(G) \rightarrow E(H)$ such that $\psi(G(e)) = uv$ if and only if $\psi_H(\varphi(e)) - \theta(u)\,\theta(v)$; such a pair (θ, φ) of mappings is called an isomorphism between $\mathbf{G}$ and H. A knot K_0 is equivalent to a knot K_n if there exists a sequence of knots $K_1, K_2, \dots, K_{n-1}$ such that K_i is an elementary deformation of K_{i-1} for $1 \le i \le n$. A prim knot is a knot that is indecomposable. Specifically, it is a non-trivial knot which cannot be written as the knot sum of two non-trivial knots [17]. Knots that are prime are said to be composite. Finally, a prime link is a link that cannot be represented as a knot sum of other link [15], [16].

MAIN RESULTS

The movement of the king (one step only in any direction) is as seen in Fig. (1) , from square (vertex) 4 to 12, then to 3 and next back to 4 . Repeating these movements give the attached graph which is isomorphic to a cycle and in return represents the trefoil knot 3_1 .

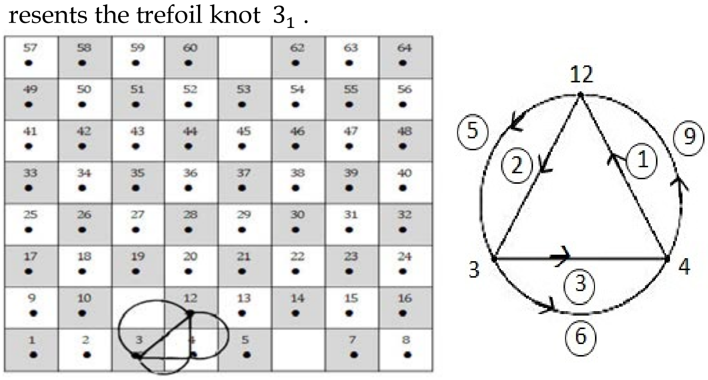


Fig. (1)

Another knot namely 4_1^2 can induced from the following: King movements: $60 \rightarrow 52 \rightarrow 51 \rightarrow 59 \longrightarrow 60$. Repeating these movements we will gain a cyclic graph which is isomorphic to knot 4_1^2. See in Fig .(2)

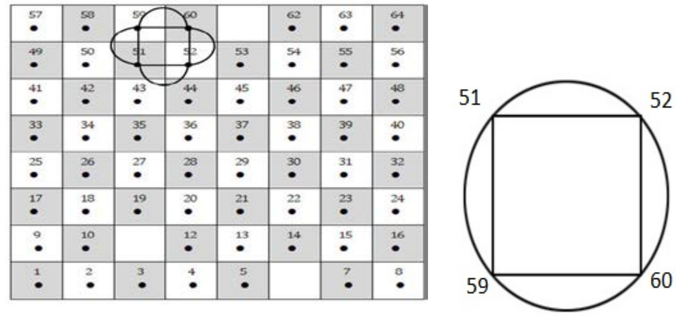


Fig. (2)

The knot 4_1 is represented by a sequence of the king movement:

$4 \rightarrow 12 \rightarrow 11 \rightarrow 4 \rightarrow 3 \rightarrow 12 \rightarrow 11 \rightarrow 34$, and then repeating these movements once; see Fig. (3).

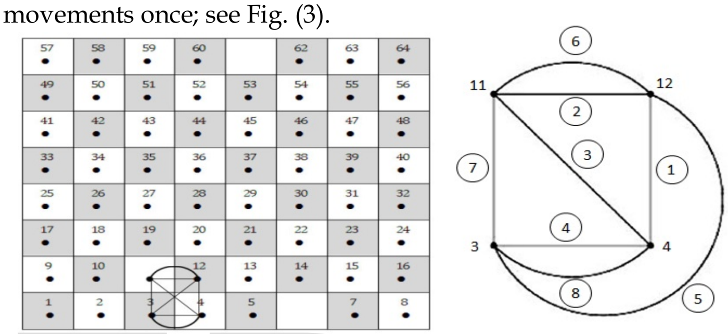


Fig. (3)

This is a cyclic graph that is isomorphic to knot 4_1 .

For knots 5_2 one can have the sequence of the king movements:

$60 \rightarrow 53 \rightarrow 44 \rightarrow 45 \rightarrow 53 \rightarrow 60 \rightarrow 52 \rightarrow 45 \rightarrow 44 \rightarrow 52 \rightarrow 60$, andthen repeating these movements; see Fig. (4).

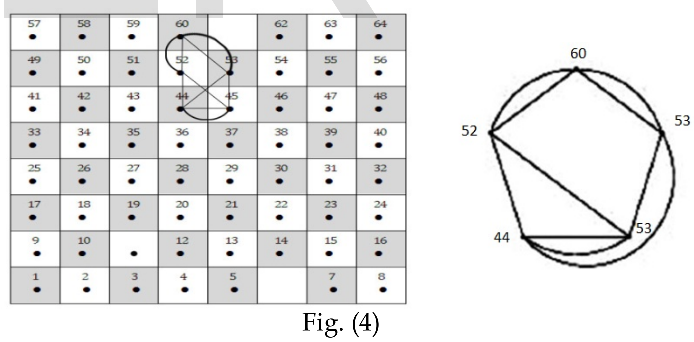


Fig. (4)

The knot 6_3 can be represented by the sequence of the king movements:

$4 \rightarrow 12 \rightarrow 13 \rightarrow 21 \rightarrow 20 \rightarrow 13 \rightarrow 4 \rightarrow 11 \rightarrow 20 \rightarrow 21 \rightarrow 12 \rightarrow 11 \rightarrow 4$, and then repeating these movements; see Fig. (5).

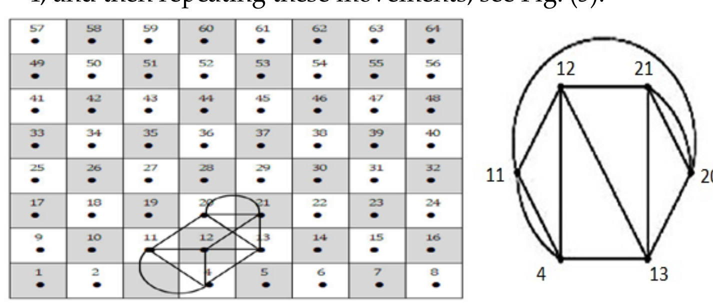


Fig. (5)

It should be noticed that if n = 2 i.e, the king moves from one square to neighbor square and return back to the original square will gain a link graph with two components. In case ≥ 3, we get the following :

THEOREM (1) :

A link graph corresponding to a knot with n-gone , n ≥ 3 can be obtained from a sequence of king's movement.

PROOF :

First of all, it should be mentioned that for n = 1,2 we don't have any knot. The simplest closed path for the king movements is 3 steps, vertical, horizontal and 45°direction in any order and then by repeating this sequence we get 3-gone link graph corresponding to trefoil knot. Suppose that, closed path for the king movement with (n-1) steps give an (n-1) gone knot. It clear that one step movement for the (n-1) gone knot contain at least one step direction45°. In this case to add another movement, we must delete one of the previous movement first and then add two movements one of them at least with 45° direction and that gives us an n- closed path and by returning back we get the required n- gone. This completes our proof.

THEOREM (2):

Any link graph without multiples (6_1^3 , 8_{18} , 10_{123}, 12_1^3, ...) cannot be obtain from any sequence of the king movement.

PROOF :

It is well known that {6_1^3 , 12_1^3, ... } forms link graphs with three components and no multiples [18], and the rest of knots which are 8_{18} , 10_{123}, ... are with one component and no multiples. In either case any given closed path will contain at least two vertices with degree less than 4. We assert that these two vertices are not located in two adjacent square in the chess, otherwise by connecting them will get a multiple which is not allowed. If the two vertices are not located in two adjacent squares, then the king has no authority to make one movement to connect them. Thus the theorem is proved.

For the movement of Bishop(moves with angle 45°only) and Rook (moves horizontally or vertically only).

Let us first discuss the Bishop movements models:

Suppose the sequence of its movements are as follows:

62 → 53 → 46 → 55 →62.Repeating these movements we will gain a cyclic graph which is isomorphic to knot 4_1^2. See in Fig. (6).

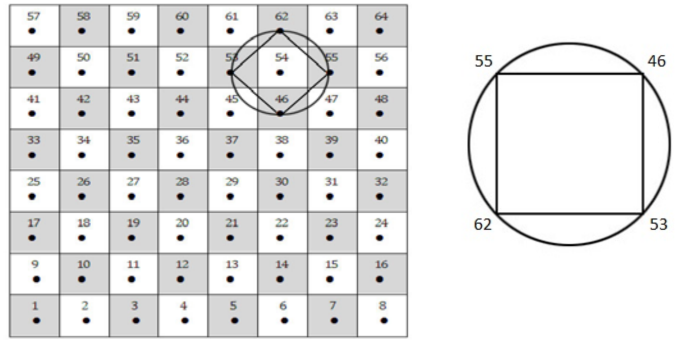


Fig. (6)

Another way of movement for the Bishop is the following: 7 → 28 → 10 → 3 → 21 → 7 → 21 → 3 → 10 → 28 → 7.Repeat this once more to get a cyclic graph which is isomorphic to 5_1 ; See Fig. (7).

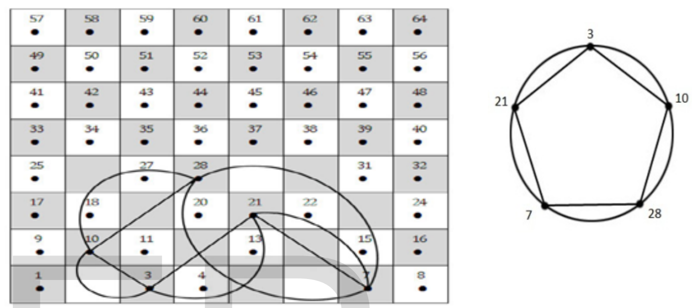


Fig. (7)

Now, let us study some models for the Rook:

Let us have the following consecutive movements: 8 → 6 → 22 → 24 → 8. Repeat them once more to gain the cyclic graphic which is isomorphic to4_1^2 ; See Fig. (8).

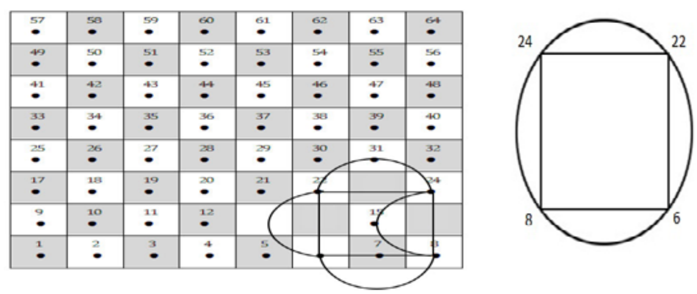


Fig. (8)

Another sequence of movement for the Rook is the following: 7 → 28 → 10 → 3 → 21 → 7. Repeating these movement we will gain a cyclic graph which is isomorphic to knot 5_1; see Fig. (9).

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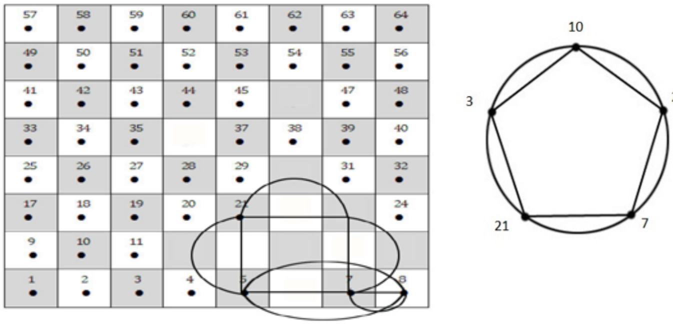

Fig. (9)

THEOREM (3) :

Any link graph with n-gone corresponds to the Rook's movement or the Rook's movements is constructed if ≥ 4.

PROOF:

It a clear that any shortest closed path can be constructed from the Rook's movements or Bishop's movements by four steps (square or rhombus) then to make a knot from any of them it needs to return it back and consequently this will construct a knot with 4- gone. This completes the proof.

For the Knight movement, It is known that its movement takes the form of letter L, this means that if Knight is in the white square, then its next move will be in a black square. So, the horse's movement will get several black and white squares. Some of its movements were analyzed as follows: $7 \to 24 \to 30 \to 13 \to 7$. And by repeating these movements we will have a cyclic graph which is isomorphic to knot 4_1^2 ; see Fig (10) .

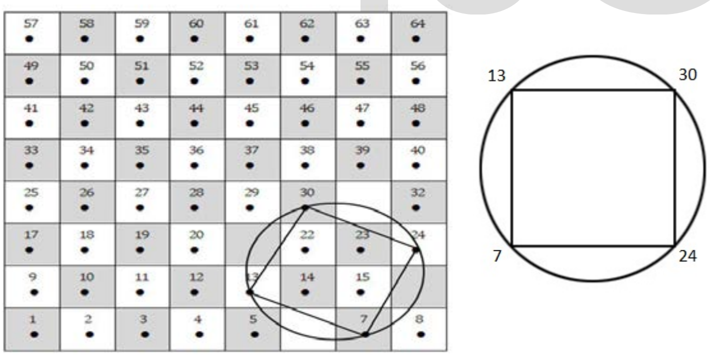

Fig. (10)

Thus we come to the following :

THEOREM(4):

Any knot with n-gone associated with the knight's movements is constructed only if n is even.

PROOF :

Since the behavior of the knight's movement takes the shape of the letter L, the first move of the knight goes from a white square to a black square or vise verse. Whiteout any loss of generality, let us suppose that its first movement from a white square to closed one. Then its next movement is from what closed, not the beginning closed square. This shows that we need another two steps to return to the original square.

For the Queen's movement, we demonstrate twelve movements' models corresponding to different kinds of knots.

The first model is the following sequence:

$5 \to 14 \to 13 \to 5$, and by repeating these one again to have a cyclic graph that is isomorphic to knot 3_1 ; see Fig. (11).

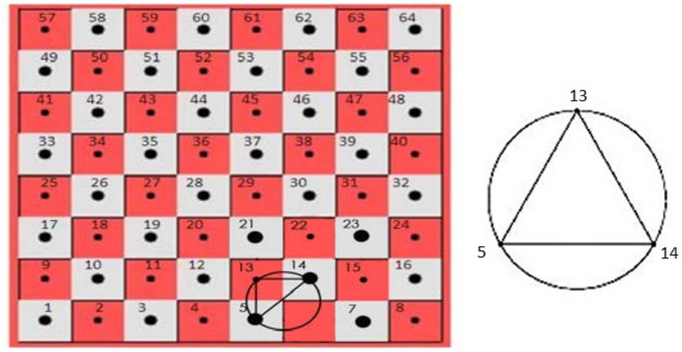

Fig. (11)

The second model is the following:

$5 \to 14 \to 21 \to 12 \to 5$. Then repeat these movement again to get a cyclic graph that is isomorphic to knot 4_1^2 ; see Fig .(12).

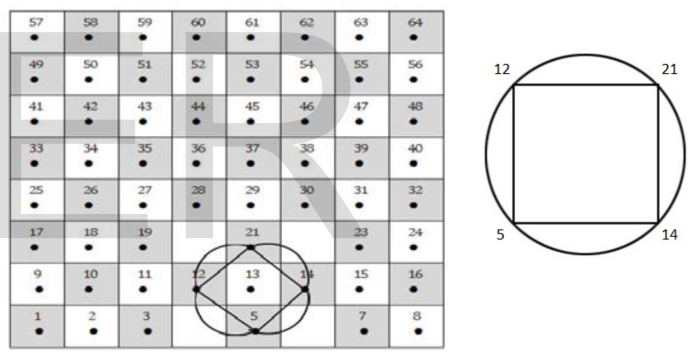

Fig. (12)

The third model is the following sequence of movements:

$5 \to 7 \to 21 \to 23 \to 7 \to 5 \to 23 \to 21 \to 5$. The induced cyclic graph will be isomorphic to knot 4_1; see Fig. (13).

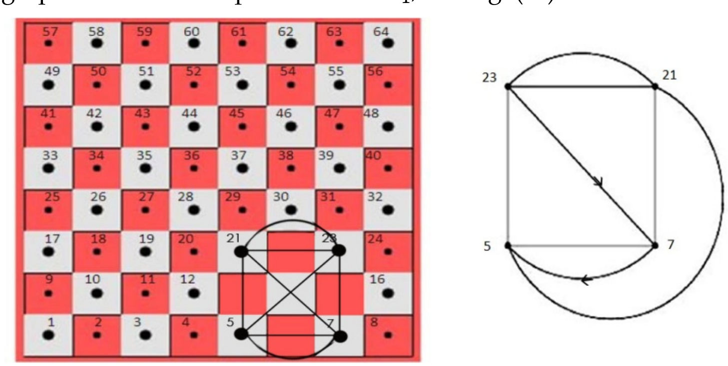

Fig. (13)

The fourth model is the following sequence of movements:

$5 \to 14 \to 30 \to 28 \to 19 \to 5$. Then repeating these one again to have a cyclic graph that is isomorphic to knot 5_1; see Fig .(14).

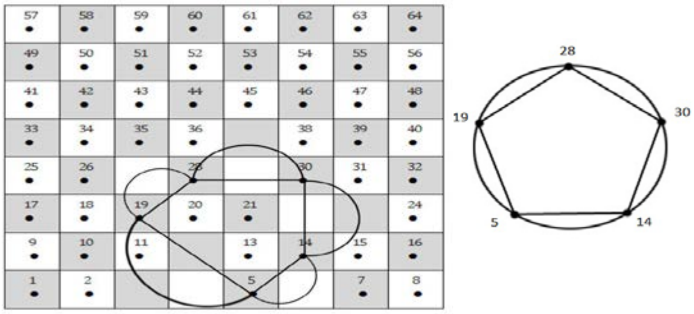

Fig. (14)

The fifth model is the following sequence of movement:
$5 \to 12 \to 21 \to 19 \to 21 \to 5 \to 3 \to 12 \to 19 \to 3 \to 5$, and then repeating these movement ; see Fig. (15).

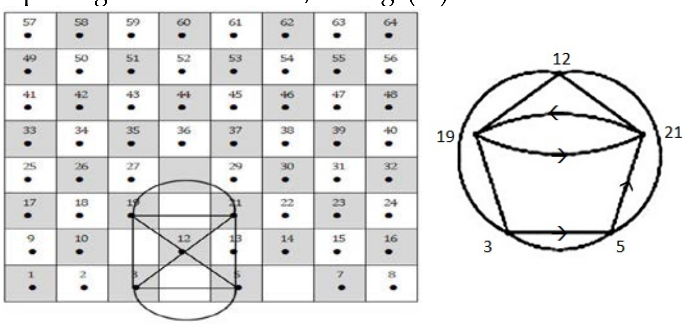

Fig. (15)

This a cyclic graph which isomorphic to knot 5_1^2 .

The sixth model is the following sequence of the movement:
$5 \to 32 \to 53 \to 32 \to 5 \to 29 \to 53 \to 26 \to 29 \to 26 \to 5$, and this repeating these movements; see Fig. (16).

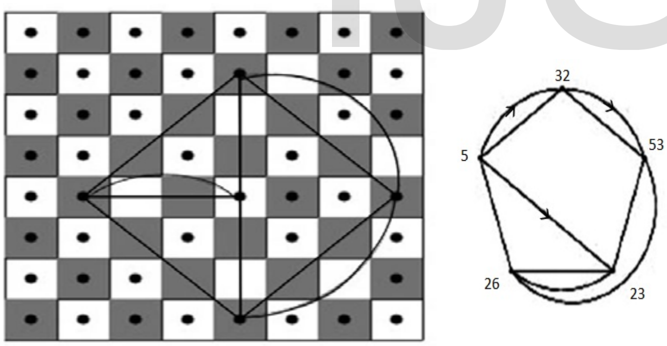

Fig. (16)

This a cyclic graph which isomorphic to knot S_2 .

The seventh model is the following sequence of the movement:
$5 \to 21 \to 23 \to 5 \to 3 \to 35 \to 21 \to 3 \to 17 \to 35 \to 17 \to 23 \to 5$, and then repeating these movements; see Fig. (17).

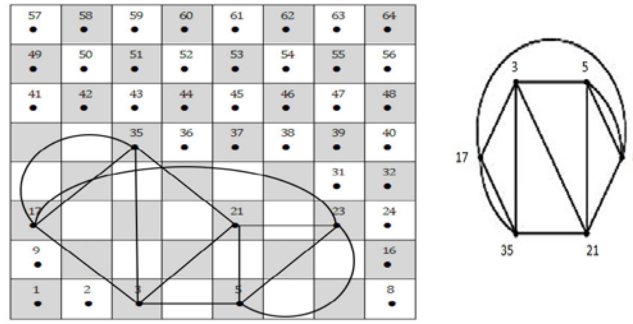

Fig. (17)

This is a cyclic graph which isomorphic to knot 6_3.

The eighth model is the following sequence of the movements:
$5 \to 23 \to 47 \to 29 \to 47 \to 23 \to 5 \to 2 \to 29 \to 26 \to 2 \to 26 \to 5$, and then repeating these movements; see Fig. (18).

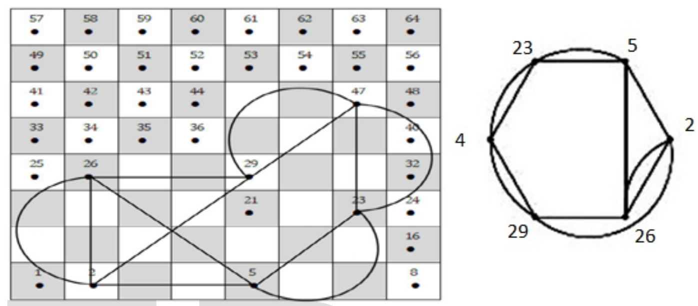

Fig. (18)

This a cyclic graph which isomorphic to knot 6_1.

The ninth model is the following sequence of the movements:
$5 \to 23 \to 37 \to 23 \to 5 \to 19 \to 37 \to 33 \to 1 \to 33 \to 19 \to 1 \to 5$, and then repeating these movements; see Fig. (19).

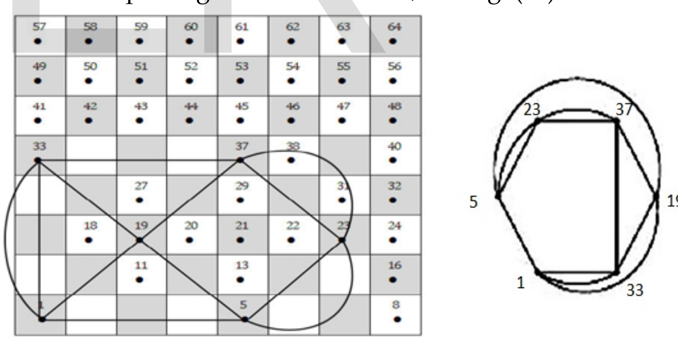

Fig. (19)

This a cyclic graph which is isomorphic to knot 6_3 .

The tenth model is the following sequence of the movements:
$5 \to 23 \to 39 \to 23 \to 5 \to 3 \to 39 \to 53 \to 35 \to 3 \to 35 \to 53 \to 5$, and this repeating these movements; see Fig .(20).

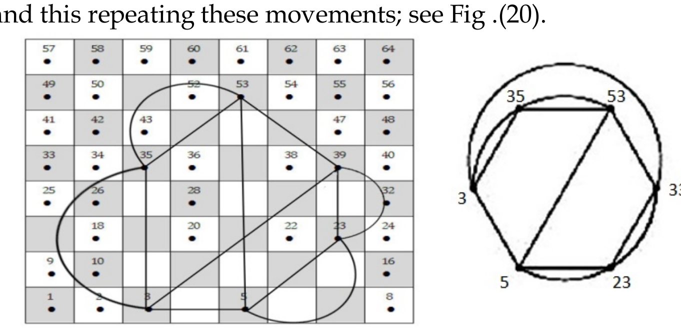

Fig. (20)

This a cyclic graph which isomorphic to knot 6_2^2 .

The eleventh model is the following sequence of the movements:
$5 \rightarrow 14 \rightarrow 30 \rightarrow 23 \rightarrow 21 \rightarrow 12 \rightarrow 30 \rightarrow 21 \rightarrow 5 \rightarrow 12 \rightarrow 14 \rightarrow 23 \rightarrow 5$, and this repeating these movements; see Fig. (21).

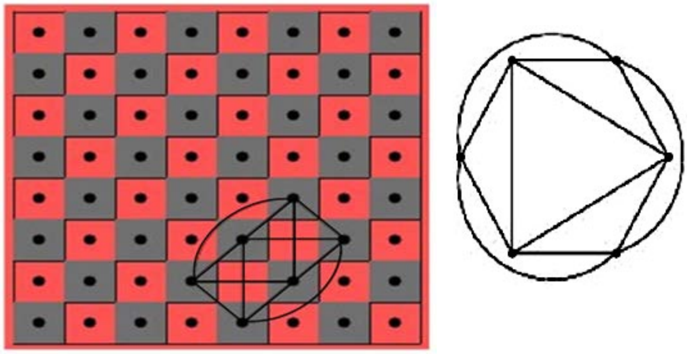

Fig. (21)

This a cyclic graph which isomorphic to knot 6_1^3.
The last model is the following sequence of the movements: $5 \rightarrow 23 \rightarrow 37 \rightarrow 19 \rightarrow 12 \rightarrow 30 \rightarrow 21 \rightarrow 14 \rightarrow 5 \rightarrow 37 \rightarrow 6 \rightarrow 14 \rightarrow 23 \rightarrow 19 \rightarrow 21 \rightarrow 12 \rightarrow 5$, and this repeating these movements; see Fig. (22).

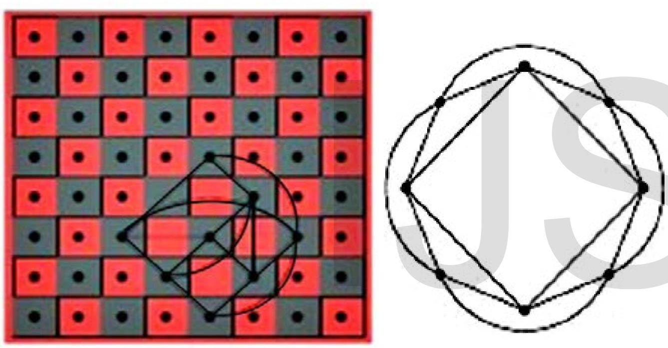

Fig. (22)

This is a cyclic graph which is isomorphic to knot 8_{18}.
Any single movement by any piece of the chessboard ,except the knight, can also be done by a single movement of the queen. One movement for the knight can be reached by two steps queen' movements to reach same position. Taking into consideration all what we have previously discussed, one can conclude the following

THEOREM (5):

Any knot can be represented by a sequence of the queen' movements.

CONCLUSION AND SUGGESTION:

For each piece in the chess except pawns ,one can form a sequence of steps to construct a particular knot. The Feynman relativistic chessboard model provides a representation of the solution to Dirc equation in (1+1)-dimension space time of quantum mechanics[ ]. A reformulation of the Feynman chessboard model has been made by Ord [ ]. The Chessboard model to a fully four-dimensional space-time of the high energy physics [ ] is still to be studied. This is extremely a new field relating the chessboard and knot theory with quantum filed theory.

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Our idea is not far from this direction

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